Risk-averse user equilibrium traffic assignment: an application of game theory

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1. Introduction

The equilibrium principle in traffic assignment ... 

- Conventional approach: safety margin from standard deviation (Uchida and Iida, 1993)
  Setting two link costs (congested and uncongested)

- Another approach: non-cooperative game
  n network users try to minimize their trip cost.
  m demons try to maximize expected travel cost.
2. Effective travel time

In conventional approach to risk assignment...

Assuming safety margin; effective travel time (Hall, 1983)

Effective travel time = scheduled arrival time – departure time
= mean travel time + safety margin

Uchida and Iida (1993), If probability for travel time is normally distributed...

Optimal Safety margin = \[
\sigma \phi^{-1}\left(\frac{\sigma}{\gamma}\right) \text{ if } \frac{\sigma}{\gamma} < (2\pi)^{-\frac{1}{2}}
\]

\(\sigma\): standard deviation of travel time distribution
\(\gamma\): penalty for late arrival
\(\phi^{-1}\): inverse of standard normal probability density function

\(\sigma \) and \(\mu\) is defined congested and uncongested link state

\[
\mu = p(v)\mu_c + (1 - p(v))\mu_u
\]
\[
\sigma^2 = p(v)^2\sigma_c^2 + (1 - p(v))^2\sigma_u^2
\]

\(p(v)\): probability of link being congested,
\(\mu_c, \mu_u\): congested and uncongested mean link travel time
\(\sigma_c, \sigma_u\): congested and uncongested standard deviation
Traffic assignment as a non-cooperative, n-player game ...
Each player (network users) wants to reduce payoff (the negative trip cost).

→ User equilibrium corresponds with mixed-strategy Nash equilibrium.
3. Equilibrium traffic assignment as an n-player game

Demonstration

At user equilibrium, minimum cost paths are used.

\[ h_j = 0 \iff g_j(h) > \min_k g_k(h) \quad \forall j \]
\[ h_j > 0 \Rightarrow g_j(h) = \min_k g_k(h) = g_{OD}(h) \]

\( h \): Flow vector, \( g_j(h) \): Route cost at route \( j \)

When \( n \) (amount of player) is large, from the Weak Law of Large Numbers...

\[ h_j \approx p_j n \]

\( p_j \): Probability that route \( j \) is chosen
3, Equilibrium traffic assignment as an n-player game

At microscopic level ... User \( a \) faces with a “game” played against other users.

Choosing and assuming a route

The strategy is ...

\[ s_a = \sum_j \pi_{aj} p_{aj} \]

\( s_a \): Mixed strategy, \( \pi_j \): pure route strategy

Users’ cost \( c_a(s) \) is ...

\[ c_a(s) = \sum_j p_{aj} c_a(s_{-a}, \pi_{aj}) \]

\( c_a(s_{-a}, \pi_{aj}) \): Cost to user \( a \) when he chooses route \( j \)

When all users minimize their trip cost ...

\[ p_{1j} = p_{2j} = \cdots = p_{nj} = p_j \]

\[ c_1(s_{-1}, \pi_{1j}) = c_2(s_{-2}, \pi_{2j}) = \cdots = c_n(s_{-n}, \pi_{nj}) \]

Here, because \( h \) is approximately equal to \( pn \),

\[ c_a(s_{-a}, \pi_{aj}) \approx g_j(h) \]
Thus,

\[ g_j(h) > g_{OD}(h) \Rightarrow p_j = 0 \iff p_{aj} = c_a(s_{-a}, \pi_{aj}) > \min_k c_a(s_{-a}, \pi_{ak}) \quad \forall a \text{ and } \forall j \]

\[ g_j(h) = g_{OD}(h) \iff p_j > 0 \iff p_{aj} > 0 \Rightarrow c_a(s_{-a}, \pi_{aj}) = \min_k c_a(s_{-a}, \pi_{ak}) \]

This means that mixed-strategy Nash equilibrium is equivalent to deterministic user equilibrium.
4. Risk-averse traffic assignment and n+1-player game

Introducing another player, “demon”: trying to maximize users’ cost by damaging link

\( G_i: \) solve simultaneously

\[
\min_{p_a} c_a(s, q) = \sum_j p_{aj} \sum_k q_k c_{ak}(s_{-a}, \pi_{aj}) \text{ for } a \in (1, \cdots, n)
\]

\[
\max_q c_{n+1}(s, q) = \sum_a q_k c_{n+1,k}(s) \text{ for demon}(n + 1)
\]

\( q: \) Vector of link damage probability
\( c_{n+1,k}(s): \) Utility of demon

In practice, finding this game’s equilibrium is difficult. However, especially when \( n \) is large, it can be found by solving following problem

\( B_1: \) solve simultaneously

\[
U: \max_q \sum_j \sum_k q_k g_{jk}(h) h_j \text{ subject to } \sum_k q_k = 1, q \geq 0
\]

\[
L: \min_h \sum_u \sum_k q_k \int_0^{v_u(h)} t_{uk}(x) dx \text{ subject to } v_u = \sum_j a_{uj} h_j, \sum_j h_j = n, h \geq 0
\]

\( t_{uk}(v_u): \) Flow dependent cost on link \( u \) in scenario \( k \),
\( a_{uj}: \) 1 if route \( j \) is used, otherwise 0
5, Risk-averse traffic assignment in a multi-community network

In several OD pair network ...

- Considering the game that \( n \) non-cooperative users try to minimize their trip cost and \( m \) OD-specific demons try to maximize users trip cost
- \( n+m \)-player game is ...

B2: for each origin-destination pair \( OD \in (1, \cdots, m) \) solve simultaneously

\[
U_{OD}: \max_{q_{OD}} \sum_j \sum_k q_{kOD} g_{jkOD}(h) h_{jOD} \quad \text{subject to } \sum_k q_{kOD} = 1, q_{OD} \geq 0
\]

\[
L_{OD}: \min_{h_{OD}} \sum_u \sum_k q_{kOD} \int_0^{v_u(h)} t_{uk}(x) dx \quad \text{subject to } v_u = \sum_j a_{uj} h_j, \sum_j h_{jOD} = n_{OD}
\]

\[
h = (h_1, h_2, \cdots, h_m), \sum_{OD} n_{OD} = n
\]
6, A solution algorithm

To solve B2, using Method of Successive Average (MSA)

Iteration counter = 1,
link flow = 0,
link failure probability = 1/(link amount)

Calculate expected link costs for each OD pair

Load the trip table for each OD pair
on the least cost path

Update the OD-specific link flow by MSA

For each OD pair, identify OD-specific failure
scenario

Convergence test

Convergence condition:

\[
\delta_q = \left| \sum_k q_k \sum_j g_{jk}(h)h_j - \max_r \sum_j g_{jr}(h)h_j \right|
\]

\[
\delta_h = \left| \sum_j h_j \sum_k g_{jk}(h)q_k - \min_r \sum_k g_{rk}(h)q_k \right|
\]

END
An example

Link cost...

\[ Cost_i = 10 + 10 \left( \frac{flow_i}{capacity_i} \right)^4 \]

- \( i \in (1, \ldots, 12) \)
- Capacity: 500 vehicles/hour
- Damage: reducing capacity to 50%
- Flow: 1000 vehicles

Fig. 1. Example network.
7. Examples

Results

- Expected trip cost: Rapidly converging

Fig. 2. Expected trip cost.
7. Examples

Results

- Link choice probability: At equilibrium, probability of all path has to be same.
- Scenario probability: Demon focuses on link 1, 3, 10, 12 and others are ignored.
8. Conclusions

- Considering risk-averse equilibrium assignment including the concept from the notion of a non-cooperative game.
- Mixed-strategy Nash equilibrium describes risk-averse flows and localities vulnerable links for each OD pair.
- Expected trip cost gives useful measure of network reliability.

- Some restrictions...
  - The demons ... forced to damage only one link a day
  - Assumption that no player and no demon knows next the moves of other players (When network users extremely pessimistic, the game becomes the Stackelberg game dominated by demons.)

Thank you for your kind attention...